Lecture 1
Conditional logics

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Classical logic

Semantic consequence

An **interpretation** of the language is a function ν from sentences to truth values obeying the usual truth conditions for propositional connectives, such as:

- $\nu(\neg A) = 1$ if $\nu(A) = 0$, and 0 otherwise;
- $\nu(A \wedge B) = 1$ if $\nu(A) = \nu(B) = 1$, and 0 otherwise.

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A is a **logical truth/tautology**, written $\models A$, iff A is true under every interpretation, equivalently, iff $\emptyset \models A$, where $\emptyset$ is the empty set (i.e. the set having no members).

Understanding the method of tableaux

A **tableau** is a proof method **by refutation** or counterexample. That is, you try to refute a sentence or provide a counterexample to an argument, and if you can't, you know the argument is valid.

Thus if we want to prove A, we do the following:

- Start a tree with $\neg A$ so that we can try to make it true, hence A false.
- Tableau rules preserve truth along one of the branches. If A is true and disjunctive, then its rule will guarantee that *at least one* of the two nodes it gives you must be true. If A is true and conjunctive, then its rule will guarantee that *both* of the nodes it gives you must be true.

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It is easiest to think of every sentence as a conjunction or a disjunction, e.g. $A \supset B$ as $\neg A \vee B$ and $\neg(A \wedge B)$ as $\neg A \vee \neg B$.

Soundness

To show that our method doesn't overgenerate, we must show that it is

Sound: it generates *only* validities. I.e. if $\Sigma \vdash A \Rightarrow \Sigma \models A$

Proof sketch. We have to prove the following: For every interpretation ν, if each node (i.e. sentence) of a branch b is true under ν and we extend b by a rule, then each node of at least one of the extended branches is true under ν.

Take the rule $\mathcal{R}^{\wedge}$ for $\wedge$. Suppose each member of b is true under ν. Then if $A \wedge B$ is a node on b, both A and B are true under ν. Hence extending b by $\mathcal{R}^{\wedge}$ results in a branch each of whose nodes is true under ν.

Completeness

To show that our method doesn't undergenerate, we must show that it is

Complete: it generates *all* the validities. I.e. if $\Sigma \models A \Rightarrow \Sigma \vdash A$.

Proof sketch. Call a branch **complete** if every rule that can be applied, has been applied. Call a tableau complete if all of its branches are complete. Then to every set of sentences there corresponds a complete tableau.

Suppose $\Sigma \nvdash A$. Then $\Sigma \cup \{\neg A\}$ corresponds to a complete tableau with open branch b. Since b is complete and open, it contains no sentence B such that both B and $\neg B$ are nodes. Define ν so that for each atom p on b, $\nu(p) = 1$, and otherwise (i.e. for atoms p not on b) let $\nu(p) = 0$. We can then verify that ν makes each node of b true, and hence each member of $\Sigma \cup \{\neg A\}$ true, and so $\Sigma \not\models A$.

Basic modal logic

Why modal logic?

For a class on *non-classical* logic, why cover modal logic? Modal logics are *extensions* of classical logic, so in that sense classical and non-rival.

The reason is that they have some of the simplest Kripke semantics, and Kripke semantics originated from work on modal logic, plus all the logics we will cover have, in some sense of the term, a Kripke semantics. In other words, non-classical operators, such as conditionals, can be viewed as modal operators.

Modal semantic consequence

Interpretations are now functions taking two inputs, a sentence and a world; the output is the same, a truth value. Modal operators are restricted quantifiers over worlds: e.g. $\Box A$ is true iff A is true at all accessible worlds.

A model has the form $\langle W, R, \nu \rangle$, where W is a non-empty set of worlds, R the accessibility relation (telling us which worlds are possible relative to others), and ν a two place interpretation function. We call the model without the interpretation, i.e. $\langle W, R \rangle$ a **frame**, and $\langle W, R, \nu \rangle$ a model **based on** the frame.

Semantic consequence, $\models$, is defined as truth preservation over *all* worlds: $\Sigma \models A$ iff A is true at any world at which every member of Σ is true.

Axiom (Hilbert) system

The weakest normal modal logic **K** is axiomatized as follows:

$$\text{All tautologies of classical logic} \qquad \text{(CL)}$$

$$\Box(A \to B) \to (\Box A \to \Box B) \qquad \text{(K)}$$

$$\frac{A \qquad A \supset B}{B} \qquad \text{(MP)}$$

$$\frac{\vdash A}{\vdash \Box A} \qquad \text{(Nec)}$$

Extensions of **K**

Here is a list of some well known axioms.

$$\Box A \to \Diamond A \tag{D}$$

$$\Box A \to A \tag{T}$$

$$A \to \Box\Diamond A \tag{B}$$

$$\Box A \to \Box\Box A \tag{4}$$

$$\Diamond A \to \Box\Diamond A \tag{5}$$

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Extensions of **K** are obtained by adding some of these axioms, and named accordingly. E.g. the well-known logic **S5**, is got by adding (T), (4) and (B) to **K**, and hence the logic is also called **KT4B**.

Conditional logics

Classical fallacies?

Classical logic validates each of the following, and many other logics validate at least some of them:

Antecedent strengthening: $A \supset B \models (A \wedge C) \supset B$

Transitivity: $A \supset B, B \supset C \models A \supset C$

Contraposition: $A \supset B \models \neg B \supset \neg A$

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But each has been claimed to be objectionable by at least some people. E.g., strengthening validates 'If it's raining then the ground is wet, so if it's raining and for some unusual reason (e.g. magic) the ground isn't wet, then the ground is wet'.

This gives us some grounds for exploring different conditionals that don't validate these inferences.

Ceteris paribus conditionals

Usually we're warranted in saying things like 'All birds fly', i.e. 'If something's a bird, then it flies', since we ignore things like dead birds, birds with clipped wings, etc. But if we mention these things we ordinary ignore, we can no longer ignore them, and a conditional once true can ring false. E.g. 'If something's a bird *and a penguin*, then it flies'.

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Let us call a clause which allows us to continue ignoring stuff—an "other things being equal" clause—a **ceteris paribus clause**. We can then continue to think of these conditionals as true even if they are not true in general. That is, we can think of conditionals as building in a cp clause. So the previous conditional becomes 'If something's a bird then—other things being equal (i.e. ignoring abnormal stuff)—it flies'.

The conditional logic C

Let us write $>$ for our new ceteris paribus conditional. Let a model in our new language have the form $\langle W, \{R_A : A \in \mathcal{F}\}, \nu \rangle$, where W is a non-empty set of worlds, ν a usual "modal" interpretation, and $\{R_A : A \in \mathcal{F}\}$ a family of accessibility relations, one for each sentence A. Think of wR_Aw' as holding when w' is like w, all things being equal concerning A (e.g. there are not dead birds around).

Then the truth conditions for $A > B$ are as follows:

$$\nu_w(A > B) = 1 \text{ iff for all } w' \text{ such that } wR_Aw',\ \nu_{w'}(B) = 1.$$

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Call the logic valid over this semantics C. It's very weak!

An equivalent semantic formulation of C

One gets logics stronger than C by putting constraints on R. Let us work, however, with an equivalent semantics. Define

- $[A] = \{w \in W : v_w(A) = 1\}$ (the truth set of A)
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Then say that

$$\nu_w(A > B) = 1 \text{ iff for all } f_A(w) \subseteq [B].$$

In other words, $A > B$ is true iff the A-selected worlds *strictly imply* B.

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The logic valid over such models $\langle W, f, \nu \rangle$ is called C^+. (I am thinking of $f_A(w)$ as taking two inputs, A and w, so more perspicuously written $f(A, w)$.)

Similarity spheres

Another way to cash out ceteris peribus clauses is in terms of a similarity metric, due to Lewis. We can think of the A-selected worlds of w as those most similar or closest to w. Then to each world we can associate a **system of spheres**, i.e. a family $\{S_0^w, S_1^w, \dots\}$ of subsets S_i^w of W *well-ordered by* $\subseteq$. Each sphere S_i^w in the system of w corresponds to a degree of similarity to w. The worlds in S_0^w are most similar to w, and worlds become less similar as we move out in the system.

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Define $f_A(w)$ as follows:

- $\emptyset$ if $[A] = \emptyset$;
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The logic valid over such models is called S.

Extensions of S

The logic C_2 is obtained from (the semantics of) S by requiring that

$$\text{if } x, y \in f_A(w), \text{ then } x = y.$$

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A weakening C_1 of C_2 which invalidates CEM is obtained from S by requiring that

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C_1 validates the contentious $A \wedge B \models A > B$.