Lecture 2
Intuitionistic logic

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Intuitionistic logic is the largest and most influential rival to classical logic. It was pioneered by L. E. J. Brouwer at the beginning of the 20th century. Today it is used in **constructive mathematics** which requires the existence of a method of constructing an object whenever that object is said to exist.

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A function satisfying certain properties can only be said to exist if there is a construction of the function, often amounting to a construction of it. E.g. the function f such that

$$f(x) = \begin{cases} 1 & \text{if Goldbach's conjecture is true;} \\ 0 & \text{otherwise} \end{cases}$$

doesn't exist constructively, assuming Goldbach's conjecture is unprovable. It exists classically, however, since Goldbach's conjecture must be either true or false.

Fitch's paradox

Another way of formulating the core intuitionistic tenet is by saying that truth is **epistemically constrained**: in order that A be true, it must be in principle be knowable.

Verificationist Principle (VP): $\forall A(A \to \Diamond KA)$

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Verificationist Principle (VP): $\forall A(A \to \Diamond KA)$

Frederic Fitch famously showed that VP entails, on plausible assumptions, that all truths are known:

Omniscience: $\forall A(A \to KA)$

This result has come to be know as **Fitch's paradox** or **the paradox of knowability**.

The BHK interpretation

The **Brouwer-Heyting-Kolmogorov interpretation** is an intuitionistically acceptable reading of the logical connectives. Here we cover only the propositional connectives.

- A proof of $A \wedge B$ consists of a proof of one conjunct followed by the other and finally $A \wedge B$.
- A proof of $A \vee B$ consists of a proof of one of the disjuncts followed by $A \vee B$.
- A proof of $A \to B$ consists of a construction converting any proof of A into a proof of B.
- A proof of $\bot$ does not exist.
- A proof of $\neg A$ consists of a proof of $A \to \bot$.

Language warning!

So as not to confuse connectives from different languages, we will use $\neg$ for intuitionistic negation, and $\sqsupset$ for the intuitionistic conditional.

Some properties of **IPL**

The following classical validities **fail** intuitionistically:

$$
\begin{array}{ll}
A \vee \neg A & \neg\neg A \to A \\
(\neg B \to \neg A) \to (A \to B) & \neg(A \wedge B) \to (\neg A \vee \neg B) \\
\neg(A \to B) \to A & (A \to B) \vee (B \to A)
\end{array}
$$

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\begin{array}{ll}
A \lor \neg A & \neg\neg A \to A \\
(\neg B \to \neg A) \to (A \to B) & \neg(A \land B) \to (\neg A \lor \neg B) \\
\neg(A \to B) \to A & (A \to B) \lor (B \to A)
\end{array}
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Intuionistic logic has the

1. **disjunction property**: if $\vdash A \lor B$ then either $\vdash A$ or $\vdash B$
2. **Glivenko property**: $\vdash A$ iff $\vdash_{\mathbf{I}} \neg\neg A$

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The disjunction property better hold given the BHK interpretation of $\vee$. The Glivenko property gives the classicist a logically equivalent **translation** of what the intuitionist is saying—in case he can't understand her!

Heredity

A model for the language has the form $\langle W, R, \nu \rangle$, where R is a **preorder** (reflexive and transitive), and ν satisfies:

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Note that while Heredity is stipulated to hold only for atoms, we can show (by induction) that it holds for all sentences.

In effect, Heredity says that we should treat atoms as having **modal force**, i.e. as having the force of $\Box p$, where $\Box$ can be read "It is provable that", since proof is truth for the intuitionist.

Truth conditions

We get the following truth conditions for the connectives:

1. $\nu_w(A \wedge B) = 1$ if $\nu_w(A) = \nu_w(B) = 1$; otherwise it is 0.
2. $\nu_w(A \sqsupset B) = 1$ if for all w' such that wRw', either $\nu_{w'}(A) = 0$ or $\nu_{w'}(B) = 1$.

Work out the truth conditions for negation, $\neg$ given our BHK clause, i.e. that $\neg A$ is equivalent to $A \sqsupset \bot$, and that $\bot$ is false everywhere.

Proof theory

Axiom (Hilbert) system

$$A \supset (B \supset A) \tag{1}$$

$$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \tag{2}$$

$$(A \wedge B) \supset A \tag{3}$$

$$(A \wedge B) \supset B \tag{4}$$

$$A \supset (B \supset (A \wedge B)) \tag{5}$$

$$A \supset (A \vee B) \tag{6}$$

$$B \supset (A \vee B) \tag{7}$$

$$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)) \tag{8}$$

$$\bot \supset A \tag{9}$$

$$\frac{A \qquad A \supset B}{B} \tag{MP}$$

Intuitionistic tableaux

In classical logic, we have $\nu(\neg A) = 1$ iff $\nu(A) = 0$, so we can represent falsity for classical tableau using syntax of the object language. In intuitionistic logic we have $\nu_w(\neg A) = 1$ implies $\nu_w(A) = 0$ (Why?), but not the converse (Why?), so we can't represent falsity at a world using our object-language connective $\neg$. What to do?

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We introduce labels into the proof language. Nodes are now not just sentences (possibly with world labels), they are sentences with labels $+$ and $-$ for expressing truth and untruth at a world. E.g. a node may now look like: $A, +i$.

The McKinsey-Tarski Translation

The **McKinsey-Tarksi** translation t is a truth-condition preserving translation from the language of intuitionistic logic to the language of modal logic:

1. $t(p) := \Box p$ for atoms p
2. $t(A \land B) := t(A) \land t(B)$
3. $t(A \lor B) := t(A) \lor t(B)$
4. $t(A \sqsupset B) := \Box(t(A) \supset t(B))$
5. $t(\neg A) := \Box\neg t(A)$

The following result follows: $\Gamma \vdash_{\mathbf{I}} A$ iff $t(\Gamma) \vdash_{\mathbf{S4}} t(A)$, where $t(\Gamma) = \{t(\gamma) : \gamma \in \Gamma\}$.

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Keep this in mind as a heuristic for the intuitionistic tableau rules: if you can remember the modal ones, it's much easier to remember the very similar intuitionistic ones.