Lecture 3
Relevant logics

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Relevant logic

Relevance

Witness some paradoxes of material implication:

$$A \supset (B \supset A);\ A \supset (\neg A \to B);\ (A \wedge \neg A) \supset B;\ A \supset (B \vee \neg B).$$

Some instances seem absurd because for of lack of relevance between antecedent and consequent, as in

> If Peter likes pretzels, then Barcelona's in Spain or it isn't.

Relevance logics originated from a desire to make right what is wrong with material implication: to formalize a conditional that does not endorse **fallacies of relevance**.

A brief history of relevance logic

The *magnum opus* of relevance logic is *Entailment* Volumes 1 (1975) and 2 (1992) of Anderson and Belnap (and numerous coauthors). They expand on the work of Wilhelm Ackermann who, in 1956, published work on a theory of **strengen Implikation**.

Relevance logic has grown into a rich and fascinating discipline. It has been mainly worked on by Australians and Americans, each working in their own style (Americans preferring many-valued world semantics, Australians two-valued world semantics). An introduction can be found on SEP (click here). But the introduction of *Entailment* Vol. 1 is an excellent—even entertaining!—read.

Anderson & Belnap on relevance

We argue below that one of the principal merits of his system of strengen Implikation is that it, and its neighbors, give us for the first time a mathematically satisfactory way of grasping the elusive notion of relevance of antecedent to consequent in "if ... then—" propositions; such is the topic of this book.
[Anderson and Belnap, 1975, p. xxi]

More Anderson & Belnap

As is well-known, this notion of relevance was central to logic from the time of Aristotle until, beginning in the nineteenth century, logic fell increasingly into the hands of those with mathematical inclinations. The modern classical tradition, however, stemming from Frege and Whitehead-Russell, gave no consideration whatever to the classical notion of relevance, and, in spite of complaints from certain quarters that relevance of antecedent to consequent was important, this tradition rolled on like a juggernaut, recording more and more impressive and profound results in metamathematics, set theory, recursive function theory, modal logic, extensional logic tout pur, etc., without seeming to require the traditional notion of relevance at all. [Anderson and Belnap, 1975, p. xxi]

Variable sharing

The exact formal properties of what constitutes a relevance logic have been debated. The following is at least agreed upon:

Variable sharing: A propositional logic **L** is relevant only if for every sentence $A \to B$ that is a theorem of **L**, A and B share at least one atom.

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For example, if $(A \wedge \neg A) \to B$ is a valid schema of classical and intuitionistic logic, then $(p_1 \wedge \neg p_1) \to p_2$ is a theorem of both. Since $p_1 \wedge \neg p_1$ and p_2 share no atom, these logics are not relevant.

Semantics

Ternary R

The most popular semantics for relevant logics employs a **ternary** accessibility relation. Why ternary? If we used a binary one in the usual way so that $A \to B$ is true (at a world) iff B is true at all accessible A-worlds, then $p \to (q \to q)$ would come out valid.

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Intuitively, $Rxyz$ says that if a conditional is true at x and its antecedent at y, then its consequent is true at z. In other words, z is the result of applying modus ponens to conditionals in x whose antecedents are in y. In yet other words, z contains the information that can be inferred by pooling (certain) information from x and y.

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You may now think about how $p \to (q \to q)$ could fail to be valid on this sketch of a semantics, but more details have to be given first to see exactly how things work. After all, we do want $q \to q$ to remain a tautology!

Normal vs non-normal worlds

Since we want to invalidate $p \to (q \to q)$ while keeping $q \to q$ valid, something tricky needs to be done. One trick is to distinguish **normal** from **non-normal** worlds.

A model will consist of a set $N \subseteq W$ of non-normal worlds at which the truth conditions for $\to$ are given by the ternary R.

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Our new truth conditions for $\to$ become:

- if $w \notin N$, then $\nu_w(A \to B) = 1$ iff for all $x \in W$, if $\nu_x(A) = 1$ then $\nu_x(B) = 1$;
- if $w \in N$, then $\nu_w(A \to B) = 1$ iff for all $x, y \in W$ such that $Rwxy$, if $\nu_x(A) = 1$ then $\nu_y(B) = 1$.

$\to$ behaves like a strict conditional at normal worlds, and "informationally" at non-normal worlds.

Negation and the Routley $*$

It turns out that we need to do something special with negation as well, as will be made clear later. One way to get the right kind of negation, called a **de Morgan negation**, is to use the so-called **Routley** $*$.

A model will now have the form $\langle W, N, R, *, \nu \rangle$ where $*$ is a function from worlds to worlds. We call $*(w)$, also written w^*, the star (world) of w.

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The logic valid over this semantics, i.e. the arguments valid over **all normal worlds**, is called B.

Proof theory

Tableaux

We need a way of keeping track of the normal/non-normal distinction. For this, sentences will now have labels of the form $+x$ or $-x$, where x can be of the form $i^{\#}$ or i. Intuitively, $i^{\#}$ is the star world of i.

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We can assume that the only normal world is labeled 0. Thus in the rule for $\mathbf{A} \rightarrow \mathbf{B}, -\mathbf{x}$, if $x = 0$, j and k must be the same.

Content inclusion

Content/informational inclusion

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Let us denote **content inclusion** by $\sqsubseteq$, which will also be a preorder (i.e. reflexive and transitive). It satisfies the following. If $w \sqsubseteq w'$, then:

1. if $\nu_w(p) = 1$ then $\nu_{w'}(p) = 1$ (Heredity)
2. $w'^* \sqsubseteq w^*$
3. if $Rw'w_1w_2$ then if $w \in N$, $w_1 \sqsubseteq w_2$, and if $w \notin N$, Rww_1w_2.

Extensions of B

The following lists constraints in R or $\sqsubseteq$ that give us extensions of B.

- (C12) If $Rabc$ then for some x such that $a \sqsubseteq x$, $Rbcx$
- (C13) If $a \in N$ then $a^* \sqsubseteq a$
- (C14) If $a \in N$ then $a^* \sqsubseteq a$; and if $a \in W - N$ then Raa^*a
- (C15) If $Rabc$ then $a \sqsubseteq c$
- (C16) If $Rabc$ then $a \sqsubseteq c$ or $b \sqsubseteq c$

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(C16) If $Rabc$ then $a \sqsubseteq c$ or $b \sqsubseteq c$

They validate, respectively:

(A12) $A \to ((A \to B) \to B)$

(A13) $A \vee \neg A$

(A14) $(A \to \neg A) \to \neg A$

(A15) $A \to (B \to A)$

(A16) $A \to (A \to A)$

References

Anderson, A. and Belnap, N. (1975).
Entailment: the logic of relevance and necessity, volume 1.
Princeton University Press.