Lecture 4
Many-valued logics

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Bivalence and contravalence

Classical semantics famously requires that interpretations be

Bivalent: for any sentence, it is either true or false

Contravalent: for any sentence, it is not both true and false

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Law of Excluded Middle (LEM): $A \vee \neg A$

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The problem is that, in a non-classical setting, LEM may hold even though bivalence fails, and LNC may hold even though contravalence fails. So these laws do not ensure the semantic contraints to which they, on the face of it, correspond.

Rejecting bivalence/gappiness

Bivalence has arguably been rejected as far back as Aristotle. Consider whether there **will be** a sea battle tomorrow. We don't know until tomorrow comes, so, one might think, it is neither true nor false *now* that there will be a sea battle tomorrow.

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Some other reasons for rejecting bivalence:

- Category mistakes (e.g. Sunday is orange and icy)
- Paradox (e.g. the liar sentence)
- Reasoning with a lack of information
- Identifying truth with proof, warranted assertibility, etc.

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Call a sentence lacking a truth value **gappy**.

Gaps and paradox

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Take the liar sentence: ‘This sentence is false’. Suppose it’s gappy. No problem—paradox solved! But what about the **strengthened liar**:

‘This sentence is not true’.

If we suppose it’s gappy, it’s not true, so what it is says is the case, so it’s true: contradiction! Supposing the liar is gappy puts us back in paradox. We need a better solution.

Gluts and paradox

This is where **dialetheism** comes in: i.e. the view that some sentences are *both* true and false, i.e. that contravalence fails.

Consider the strengthened liar again, and call it L. If we suppose L is true, false or gappy, we're back in paradox (i.e. that it's both true and false). Seems like there's no other solution than to accept that it's both true and false.

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Of course there are other options, like denying the so-called T-rules:

T-IN: from A, infer TA ('It is true that 'A'')

T-OUT: from TA, infer A

But it seems definitive of truth that the T-rules out. Upon further reflection, dialetheists argue, the dialetheist solution to the paradoxes looks most compelling.

Semantics

Strong 3-valued Kleene logic

One of the most famous three-valued logics is *Strong Kleene logic*, the three-valued variant we'll refer to as $\mathbf{K_3}$.

Compound sentences are assigned truth values in basically exactly the same way as in classical logic except that some atomic sentences can be gappy, resulting in gappy compounds.

$\wedge$	t	f	
t	t	f	
f	f	f	f
		f	

	$\neg$
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This yields a **partial two-valued** interpretation of $\mathbf{K_3}$. We will now look at its more more popular **3-valued** interpretation.

The 3-valued interpretation of $\mathbf{K_3}$

Instead of valuations that may assign no value to sentences, i.e. **partial valuations**, three-valued **total valuations** are more popular. Let n be the truth value ‘neither truth nor falsity’.

$\wedge$	t	f	n
t	t	f	n
f	f	f	f
n	n	f	n

	$\neg$
t	f
f	t
n	n

Clearly we’ve just filled in the previous slide’s tables’ blank spaces with ‘n’.

Validites and consequence

In many-valued logics, since there are more than two truth values, there may be more than one **truth-like value**. We call such values **designated**. With $\mathbf{K_3}$ there is only one designated value, like classical logic, truth.

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Now consider which sentences are valid in $\mathbf{K_3}$ (i.e. sentences which follow from no premises). **None!**

Many-valued models, more formally

A model has the form $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\} \rangle$ for a many-valued logic is a triple where

1. $\mathcal{V}$ is a non-empty set of truth values
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In the case of $\mathbf{K_3}$, $\mathcal{D} = \{1\}$, $\mathcal{V} = \{1, i, 0\}$ and interpretations satisfy:

- $\nu(A \wedge B) = 1$ iff $\nu(A) = \nu(B) = 1$;
- $\nu(A \wedge B) = 0$ iff $\nu(A) = 0$ or $\nu(B) = 0$;
- $\nu(A \wedge B) = i$ otherwise;
- $\nu(\neg A) = 1$ iff $\nu(A) = 0$;
- $\nu(\neg A) = 0$ iff $\nu(A) = 1$;
- $\nu(\neg A) = i$ otherwise.

Weak 3-valued Kleene/Bochvar logic

Another three-valued non-bivalent logic is **weak** 3-valued Kleene logic. Unlike $\mathbf{K_3}$, we have that a sentence takes the value i whenever *any part of it takes* i.

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One interpretation of this logic is a "garbage in, garbage out" or "nonsense in, nonsense out" (Bochvar). If A is nonsense, e.g. 'Green ideas sleep furiously', then so is $A \wedge B$ for any B.

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Weak Kleene logic is so-called because it is obviously much weaker than Strong Kleene logic: the weak logic is a sublogic of the strong one. E.g. $A \models A \vee B$ is valid in the strong logic, but not the weak one. (Why?)

Glutty logics

The Logic of Paradox, **LP**

Assume that instead of interpreting i as a gap, we interpret it as a **glut**. But then taking the value i means being both true and false, and hence true, and hence **designated**. So we need to add i to $\mathcal{D}$.

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Another oddity is that **LP** has no conditional! For $\supset$ does not even satisfy modus ponens, a rule thought *constitutive* of the conditional.

3-valued Rule Mingle, $\mathbf{RM_3}$

One way to fix the lack of conditional in **LP** is to replace its conditional, definable as $\neg A \vee B$, with another not so definable:

$\supset$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

Intuitionistic and many-valued logic

A logic is **finitely many-valued** if it is characterized by some $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\} \rangle$ where $\mathcal{V}$ is finite.

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The following is an interesting result of Gödel:

Intuitionistic logic is not a finitely many-valued logic.

Proof of Gödel's result

Proof. Suppose **I** is a n-valued logic. Since $A \leftrightarrow A$ is **I**-valid, if A and B have the same truth value, $A \leftrightarrow B$ must have the same truth value. Since there are only n values, the following sentence constructed out of $n+1$ atoms is valid:

$$(p_1 \leftrightarrow p_2) \vee \cdots \vee (p_1 \leftrightarrow p_n) \vee (p_2 \leftrightarrow p_3) \vee \cdots \vee (p_n \leftrightarrow p_{n+1})$$

It says that at least two of the atoms share their truth value.

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Since there are $n + 1$ of them, this must be so under any assignment of values to atoms, since there are only n values. But since **I** has the disjunction property, it follows that one of the disjuncts is valid; say it is $p_i \leftrightarrow p_j$. Since $i \neq j$ (given the construction of the disjunction), there is an assignment giving p_i and p_j different values, making $p_i \leftrightarrow p_j$ false. Contradiction.

A Gödel-like result for relevance logic **R**

Just as with intuitionistic logic, we have that

Relevance logic **R** is not a finitely many-valued logic.

Proof. One can show that R, i.e. all its valid arguments and rules, is valid over a certain infinitely many-valued matrix $\mathcal{M}$ (p. 219 of Priest). Thus if something is not valid over $\mathcal{M}$, it can't be valid in **R**.

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One can then show that no disjunction of all sentences of the form $p_i \leftrightarrow p_j$, for all i and j such that $0 \leq i \leq j \leq n$, is valid over $\mathcal{M}$. It follows that no such disjunction is valid in **R**.

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But if **R** were finitely many-valued, one such disjunction would be valid in **R**. (One would need to show this.) So **R** is not finitely many-valued.